Orthogonality Relations for Group Characters

Proofs of the orthogonality relations for characters of group representations often seem computational and unenlightening, even from authors like J.P. Serre and Michael Artin who are known for the quality of their exposition. I learned this proof from a paper of Raoul Bott (writing about representations of compact Lie groups), and I think it deserves to be better known.

1. Let G be a finite group with n elements. We will view a representation of G as as "**G-module**," that is, a vector space U (over a field K, which we will generally take to be $\mathbb{C}$) equipped with a (left) action of G on U. Equivalently, it can be viewed as a homomorphism T mapping G to the group of invertible linear transformations of U; we will use these two points of view interchangeably. (It can also be viewed as a module over the group ring $K[G]$, but we will say no more about that.) A **G-map** from a G-module V to a G-module W is just a linear map L which respects the G action, that is $L(gv) = gL(v)$. A **G-isomorphism** is a G-map which is an invertible linear transformation; its inverse will also be a G-map.

2. If U is a G-module, let U^G denote the subspace of **G-invariant** elements of U, that is, the set of all $u \in U$ such that $gu = u$ for all $g \in G$. Define the **averaging map** $A_U \colon U \to U$ by $u \mapsto \frac{1}{n}\sum_{g\in G} gu$. (This does not work when the characteristic of K divides n, but it works fine when $K = \mathbb{C}$.) When $u \in U^G$, $A_U(u) = u$. It is not hard to see that $gA_U(u) = A_U(u)$, so that $A_U(u) \in U^G$. As a result $A_U = A_U^2$ is a projection from $U \to U^G$. (It is also not hard to see that $A_U(gu) = A(u)$, from which it follows that A_U is a G-map.) The trace $\operatorname{tr} A_U$ is the dimension of U^G, since, in an appopriate basis, a projection map has only 0s and 1s on the diagonal.

3. If U is a G-module, the **character** of U is a mapping $\chi_U \colon G \to K$ which maps $g \in G$ to the trace of $T(g)$. In this notation, the trace of the averaging map A_U is equal to $\frac{1}{n}\sum_{g\in G}\chi_U(g)$.

4. If V is a G-module, we can make the dual of V, V^*, into a G-module by defining $g\mu = \mu \circ T(g^{-1})$. (It needs to be g^{-1} to make it a *left* action by G.) When $K = \mathbb{C}$, the character $\chi_{V^*}(g) = \overline{\chi_V(g)}$. To prove this, note that any eigenvalue λ of $T(g)$ must be a root of unity, since G has finite order. Thus $\lambda^{-1} = \overline{\lambda}$.

5. If V and W are G-modules, we can make the tensor product $V \otimes W$ into a G-module such that $g(v \otimes w) = (gv) \otimes (gw)$. The character $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.

6. If V and W are G-modules, we can make the vector space of linear transformations $V \to W$, $\operatorname{Hom}(V, W)$, into a G-module by defining $gL = T(g) \circ L \circ T(g^{-1})$ for $L \in \operatorname{Hom}(V, W), g \in G$. The invariant elements $\operatorname{Hom}(V, W)^G$ are precisely the G-maps $V \to W$.

7. The tensor product $V^* \otimes W$ is G-isomorphic to $\operatorname{Hom}(V,W)$. Under this isomorphism, $\mu \otimes w$ maps to the linear transformation which takes $v \mapsto \mu(v)w$. If μ is represented by the row-vector $\vec{M}$ and w is represented by the column vector $\vec{W}$, this linear transformation is represented by the matrix $\vec{W}\vec{M}$, the $m \times n$ product of an $m \times 1$ matrix by a $1 \times n$ matrix. (How do m and n relate to the dimensions of V and W?)

8. When V and W are irreducible representations, Schur's lemma tells us that when V is not isomorphic to W, any G-map $V \to W$ must be equal to 0. In this case $\dim \operatorname{Hom}(V,W)^G = 0$. When V is isomorphic to W and K is algebraically closed (as $\mathbb{C}$ is), Schur's lemma tells us that $\dim \operatorname{Hom}(V,W)^G = 1$.

9. If V and W are irreducible representations, it follows that

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\begin{aligned}
\frac{1}{n}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g) &= \frac{1}{n}\sum_{g\in G} \chi_{V^*}(g)\chi_W(g) && \text{by (4)} \\
&= \frac{1}{n}\sum_{g\in G} \chi_{V^*\otimes W}(g) && \text{by (5)} \\
&= \operatorname{tr} A_{V^*\otimes W} && \text{by (3)} \\
&= \operatorname{tr} A_{\operatorname{Hom}(V,W)} && \text{by (7)} \\
&= \dim \operatorname{Hom}(V,W)^G && \text{by (2)} \\
&= \begin{cases} 1 & \text{if } V \text{ is isomorphic to } W \\ 0 & \text{otherwise} \end{cases} && \text{by (8)}
\end{aligned}
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